Stochastic Bound on Delay for Guaranteed Rate Nodes

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Abstract— **We find that the probabilistic bound on delay in [1] is incorrect. The problem originates from: (1) the difference between stationary and Palm probabilities, and (2) treating the arrival counting process over some random time intervals as if the time interval would be fixed. The error is propagated to some later work, e.g. [2], which derived some probabilistic delay bounds based on [1]. We give fixes to the above problem.**

Keywords— **Fair queueing, guaranteed rate clock, packet scheduling, quality of service, Palm probability.**

I. Introduction

IN [1] the authors derive a probabilistic bound on delay through a sequence of guaranteed rate (GR) nodes[1] [3], under the assumption that the arrival process is with Exponentially Bounded Burstiness (E.B.B.) [4]. In [2], the authors propose a credit-based fair scheduler, show that it belongs to the class of GR nodes, and then directly apply the probabilistic bound on delay found in [1]. More generally, it is known that many schedulers can be described as GR nodes, with appropriately defined rate and latency parameters. Further, the concept of GR node (recalled below) is a convenient way to abstract the main properties of a complex system, such as a router or a subnetwork, which is made of schedulers and delay elements, work conserving or not (Chapter 2 in [5]). Thus, it is important to have delay bounds for GR nodes. The concept of GR node is, roughly speaking, equivalent to a service curve concept [5].

However, we show that the probabilistic bound on delay in [1] is incorrect (Theorem 4 therein). The error is propagated to Theorem 2 and Theorem 4 of [2]. We note that the reference [1] can be also found as [3], which suffers from the same problem.

We first introduce some notation and then explain where the problem comes from. We assume that the arrival of packets is described by a stationary marked point process $(T_n, L_n)_{n\in\mathbb{Z}}$, where T_n is the arrival time of packet n and L_n its size in bits. Call $L^{\max}$ the maximum packet size. For an interval $\mathcal{I} \subset \mathbb{R}$, let $A\mathcal{I}$ be the number of bits observed in $\mathcal{I}$, i.e., the number of bits in $\mathcal{I}$ of those packets with their last bits falling in $\mathcal{I}$; we assume that $A(0,t]$ is a cádlág function of t (i.e., is right-continuous with left-hand limits). We denote with P_A the Palm probability associated with A (the probability given that $A\{0\} > 0$, i.e., there exists some n such that $T_n = 0$). Likewise, E_A is the expectation with respect to P_A. See, for instance, [6] for an exposition of Palm calculus.

A is said to be with (ρ, C, c)-E.B.B., if for all $\sigma \geq 0$ and $0 \leq s < t$,

$$\mathsf{P}(A[s,t] \geq \rho(t-s)+\sigma) \leq Ce^{-c\sigma}. \tag{1}$$

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[1] The original denomination in [3] is Guaranteed Rate Clock *scheduler*. We use the phrase "GR node" instead, as the GR property applies more generally to complete systems which cannot be defined as single schedulers.

In (1), the probability is with respect to the stationary distribution of A.

We recall the definition of a GR node in [3]. A system is a GR node with rate R and latency E it satisfies the following. First, there exists a sequence V_n (the "virtual finish times") which satisfies the recursion

$$V_n = \max[V_{n-1}, T_n] + \frac{L_n}{R},\ n \in \mathbb{Z},$$

and such that for all $n \in \mathbb{Z}$, there exists some $m \leq n$ with $V_m \leq T_m$. Second, the departure time process $(T'_n)_n$ satisfies $T'_n \leq V_n + E$ for all n.

Note that our definition is slightly more general than the original definition in [1], in order to fit a stationary framework. It is not difficult to observe that the GR node definition is equivalent to saying that, for all $n \in \mathbb{Z}$, there exists some $m \leq n$ such that

$$T'_n \leq T_m + \frac{L_m + \cdots + L_n}{R} + E. \tag{2}$$

At some point (proof of Theorem 4 in [1]), the authors consider the following event. Fix some n, then consider

$$\{A[T_{N_n}, T_n] \geq \rho(T_n - T_{N_n}) + \sigma\}, \tag{3}$$

where N_n is the largest integer not greater than n, for which $V_{N_n-1} < T_{N_n}$.

From the assumption that A is with (ρ, C, c)-E.B.B., the authors in [1] conclude:

$$\mathsf{P}(A[T_{N_n}, T_n] \geq \rho(T_n - T_{N_n}) + \sigma) \leq Ce^{-c\sigma}.$$

However, we find that this does not follow from the E.B.B. assumption. The reason is twofold. First, the above event considers the number of bits observed over an interval, given there is an arrival at the boundary of the interval; thus the underlying probability is the Palm probability and not the stationary probability as given in E.B.B. definition. Second, the length of the interval over which bits are observed is random, not fixed.

One may intuitively think of $A[T_{N_n}, T_n]$ as the number of bits one would observe if one picks up at random an arrival packet n, and then counts the number of bits observed since the beginning of the current busy period up to the time instant T_n. This is rigorously true if $E = 0$ (then, $T'_{N_n-1} < T_{N_n}$).

We show now that this methodological error has a fatal consequence on the validity of the final result in [1] (namely, Theorem 4). We do this by exhibiting an example where the delay bound in [1] does not hold.

Example 1 (M/D/1/∞) Consider a FIFO work-conserving server with service rate ρ; clearly, this system is a GR node with rate ρ and latency 0. The arrival process is Poisson with intensity $\bar{\rho}$ and packets are unit-length. For stability, we require $\bar{\rho} < \rho$.

First, we show that A is an E.B.B. process. Note that $\mathsf{E}(e^{\theta A[0,t]}) = e^{\bar{\rho}t(e^{\theta}-1)}$. It is routinely observed that, for $\rho > \bar{\rho}$,

$$\mathsf{E}(e^{\theta A[0,t]}) \le e^{\rho t \theta},$$

for any $\theta \in [0, \theta_0]$, where θ_0 is the unique strictly positive solution of

$$e^{\theta_0} - 1 = \frac{1}{\lambda}\theta_0.$$

By Chernoff's bound, one then obtains that A is $(\rho, 1, \theta_0)$-E.B.B., i.e., for any $t \ge 0$,

$$\mathsf{P}(A[0,t] - \rho t \ge \sigma) \le e^{-\theta_0 \sigma}. \quad (4)$$

Call D_0 the delay incurred by an arbitrary packet labeled with 0. Then, we should be able to apply the result in [1], which, here, translates to:

$$\mathsf{P}_A(D_0 \ge u) \le e^{-\theta_0 \rho u} \quad (5)$$

Second, we directly compute the delay distribution and match it against the hypothetical bound in (5).

Let $Q(t)$ be the unfinished work at t. Then, we know (see for instance [7], Sec. 6.1.1, p. 112), for $m \in \mathbb{Z}^+$,

$$\mathsf{P}(Q(0) > m) = 1 - (1-\lambda)\sum_{n=0}^{m} \frac{[\lambda(n-m)]^n}{n!} e^{-\lambda(n-m)}, \quad (6)$$

where by definition $\lambda = \frac{\bar{\rho}}{\rho}$.

We consider the event $\{D_0 \ge u\}$, where, for simplicity, we require ρu to be an integer. Thus, $u \in \{\frac{1}{\rho}, \frac{2}{\rho}, \cdots\}$. We now prove that

$$\mathsf{P}_A(D_0 \ge u) = \mathsf{P}(Q(0) > \rho u - 2). \quad (7)$$

To show this, first note $\mathsf{P}_A(D_0 \ge u) = \mathsf{P}_A(Q(0) \ge \rho u)$. By PASTA (Poisson Arrivals See Time Averages), we have $\mathsf{P}_A(Q_-(0) \ge \rho u) = \mathsf{P}(Q(0) \ge \rho u)$; see, e.g., [6], Sec. 3.1, p. 165. Second, observe that $\mathsf{P}_A(Q(0) \ge \rho u) = \mathsf{P}(Q(0) \ge \rho u - 1) = \mathsf{P}(Q(0) > \rho u - 2)$, which shows the stated equality.

We can now do a direct evaluation of (7), using a numerical computation of (6), and compare it against $e^{-\theta_0 \rho u}$, the upper bound predicted by (5). We see in Fig. 1 that the bound does not hold, i.e. $\mathsf{P}_A(D_0 \ge u) > e^{-\theta_0 \rho u}$.

Comment. The example demonstrates that the probabilistic delay bound in [1] is incorrect. Notice that, in this example, given that the arrival process is Poisson, there is no bias due to the difference between Palm and stationary probabilities (PASTA). It will be seen later that when PASTA does not hold, we expect the delay bound to be even larger.

The rest of the note is organized as follows. In Section II we give a correct probabilistic bound on delay through an isolated GR node for an E.B.B. arrival process. By application of known concatenation properties, we extend the result to a sequence of GR nodes. In Section III we give fixes to the Theorems in [1] and [2].

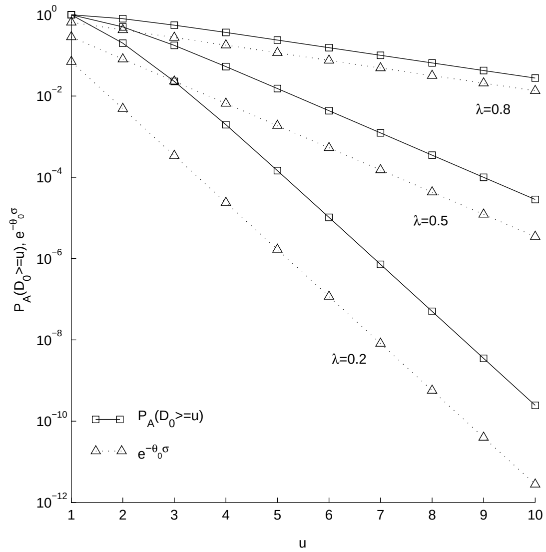

Fig. 1. Numerical values show $\mathsf{P}_A(D_0 \ge u)$ (7) is greater than $e^{-\theta_0 \rho u}$ ($\rho = 1$).

II. Probabilistic Bound on Delay For GR Nodes

A. Single Node Case

Consider a single isolated GR node with rate R and latency E. From the definition of GR node (2), it follows that for all $n \in \mathbb{Z}$, there exits some $m \le n$ such that:

$$D_n \le \frac{L_m + \cdots + L_n}{R} - (T_n - T_m) + E.$$

Thus,

$$D_n \le \max_{m \le n}[\frac{A[T_m, T_n]}{R} - (T_n - T_m)] + E. \quad (8)$$

Define $\tilde{Q}(t) = \sup_{s<t}[A(s,t] - R(t-s)]$. $\tilde{Q}$ is the unfinished work of a hypothetical work-conserving constant rate server with rate R, fed with the same arrival process A as our original system. Now, from (8), for an arbitrary packet labeled with 0, we have

$$\mathsf{P}_A(D_0 \ge u) \le \mathsf{P}_A(\tilde{Q}(0) \ge R(u-E)),\ u \ge 0. \quad (9)$$

Our aim is to obtain a bound on $\mathsf{P}_A(D_0 \ge u)$ in terms of the stationary distribution of $\tilde{Q}$. To that end, we use a corollary of Theorem 3 in [8] (distributional Little's law) that we pose as a lemma. Call $\tilde{Q}_-(t)$ the limit to the left of $\tilde{Q}$ at time t.

Lemma 1 ([8]) Consider a work-conserving constant service rate server with rate R. The server is fed with packetized A; supposed to be stationary random measure with intensity $\bar{\rho} < R$. Then, for any non-decreasing measurable function $\psi : \mathbb{R}^+ \to \mathbb{R}$,

$$\left(1 - \tfrac{\bar{\rho}}{R}\right)\psi(0) + \tfrac{\bar{\rho}}{R}\mathsf{E}_A[\psi(\tilde{Q}_-(0))] \le \mathsf{E}[\psi(\tilde{Q}(0))] \le$$
$$\le \tfrac{\bar{\rho}}{R}\mathsf{E}_A[\psi(\tilde{Q}(0))] + \left(1 - \tfrac{\bar{\rho}}{R}\right)\psi(0).$$

Note that

$$\tilde{Q}(0) = \tilde{Q}_-(0) + A\{0\} \le \tilde{Q}_-(0) + L^{max}.$$

Thus, for $u \ge 0$,

$$\mathsf{P}_A(\tilde{Q}(0) \ge R(u-E)) \le \mathsf{P}_A(\tilde{Q}_-(0) \ge R(u - E - \frac{L^{max}}{R})).$$

Next, use the right hand-side in (9), and then apply Lemma 1 with $\psi(x) = 1_{x \geq R(u-E-\frac{L^{max}}{R})}$, for $u > E + \frac{L^{max}}{R}$. As a result, we obtain

$$\mathsf{P}_A(D_0 \geq u) \leq \frac{R}{\bar{\rho}} \mathsf{P}(\tilde{Q}(0) \geq R(u - E - \frac{L^{max}}{R})^+). \quad (10)$$

Note that we are allowed to add $(\cdot)^+ = \max[\cdot, 0]$ above simply because $\tilde{Q}$ is a non-negative process.

Comment. Notice that the assumption that the arrival process A is (ρ, C, c)-E.B.B. implies that $\bar{\rho} \leq \rho$ [4]. As an aside, note that $R(u - E - \frac{L^{max}}{R})^+$ is the minimum service curve offered by a GR node with rate R and latency E (Corollary 2.1.1. [5]).

We next show the main result of this section – probabilistic bound on delay through a GR node for E.B.B. arrival process.

Theorem 1: Consider a GR node with rate R and latency E. The node is fed with stationary random A of intensity $\bar{\rho}$; in addition, A is (ρ, C, c)-E.B.B. with $\rho < R$. If time is continuous, then, for $u > 0$,

$$\mathsf{P}_A(D_0 \geq u + E + \frac{L^{max}}{R}) \leq \frac{R}{\bar{\rho}} \frac{Ce^{c\rho\delta}}{1 - e^{-c(R-\rho)\delta}} e^{-cRu}, \quad (11)$$

where $\delta > 0$ is such that

$$\frac{Ce^{c\rho\delta}}{1 - e^{-c(R-\rho)\delta}} e^{-cR\delta} \geq 1.$$

If time is discrete, then, for $u > 0$,

$$\mathsf{P}_A(D_0 \geq u + E + \frac{L^{max}}{R}) \leq \frac{R}{\bar{\rho}} \frac{C}{1 - e^{-c(R-\rho)}} e^{-cRu}. \quad (12)$$

Comment. We first compare (11) with [4]. The bounds in [4] are for the unfinished work, or *virtual waiting time* of a work-conserving constant service rate server; they are for the steady-state probability P. Their validity requires only to suppose that A is with E.B.B.; we do not need the additional assumption that A is stationary. In contrast, for the bound on *waiting time* distribution given here (which is for the Palm probability P_A), we need to suppose that A is stationary, in order to apply the result of [8]. The difference between the steady-state and Palm probabilities is reflected in the pre-factor $\frac{R}{\bar{\rho}}$ in (11) and (12).

Next, we discuss how (11) differs from Theorem 4 in [1] (equation (46) therein). The discussion is for an isolated GR node; we later give extension for the delay through a sequence of schedulers. First, note there is an additional latency term $\frac{L^{max}}{R}$. Second, we have the additional pre-factor $\frac{R}{\bar{\rho}}$, as mentioned earlier. Third, we have an additional pre-factor $\frac{Ce^{c\rho\delta}}{1-e^{-c(R-\rho)\delta}}$. In total, we expect (11) to be larger than the (incorrect) bound in [1].

Proof: The proof follows from (10) and a known bound for the unfinished work of a constant rate server with rate R and E.B.B. arrival process (see. Equation (4) in [4]). Note that the condition on δ reads as

$$0 < \delta \leq \frac{\ln(1 + C)}{c(R - \rho)}.$$

(12) is proved likewise (see remark to Theorem 1 in [4]). ■

B. End-to-end Delay Bound

Consider a sequence of H GR schedulers with rates and latencies $(R_h, E_h)_{h=1}^H$. It is known that this concatenation is a GR node with rate $R = \min_{1 \leq h \leq H} R_h$ and latency $E = \sum_{h=1}^H E_h + L^{\max} \sum_{h=1}^{H-1} \frac{1}{R_h}$ ([1]; see also Sec. 2.1.3 of [5]). Suppose the sequence of GR schedulers is fed with a (ρ, C, c)-E.B.B. arrival process A with $\rho < R$. Then, a probabilistic bound on the end-to-end delay is obtained from Theorem 1 by replacing R and E as defined in this section.

III. Correction of the Theorems in [1] and [2]

We first give a correct version of Theorem 4 in [1].

Theorem 2: If flow f, with stationary random arrival process A^f, conforms to E.B.B. with parameters $(\rho_f, \Lambda_f, \gamma_f)$, intensity $\bar{\rho}^f$, and the scheduling algorithm at each of the servers on the path of a flow belongs to GR for the flow (rate R^f and latency E_h^f, $h = 1, \ldots, H$, such that $\rho^f < R^f$), then the end-to-end delay of packet 0 denoted as D_0^f is given by:

$$\begin{aligned} &\mathsf{P}_{A^f}(D_0^f \geq u + H\tfrac{L^{max}}{R^f} + \textstyle\sum_{h=1}^H E^h) \leq \\ &\leq \tfrac{R^f}{\bar{\rho}^f} \tfrac{\Lambda_f e^{\gamma_f \rho^f \delta}}{1-e^{-\gamma_f(R^f-\rho^f)\delta}} e^{-\gamma_f R^f u},\ u > 0, \end{aligned} \quad (13)$$

where $\delta > 0$ is such that

$$\frac{\Lambda_f e^{\gamma_f \rho^f \delta}}{1 - e^{-\gamma_f(R^f-\rho^f)\delta}} e^{-\gamma_f R^f \delta} \geq 1.$$

Above L^{max} is the maximum packet length of the flow f.

Comment. In [1], in their notation, the authors use $\max_{1 \leq m \leq j} L_m$ in place of $L^{\max}$ in (13). This is justified for the deterministic delay bound of the j-th packet [1]. In the probabilistic setting, $\max_{1 \leq m \leq j} L_m$ is indeed itself random. Given that we look at the delay of an arbitrary packet j, one would need to use $L^{\max}$ instead. Note also that for stability, we need to require $\rho^f < R^f$; compare this with [1], where $\rho^f = R^f$.

Next we give fixes to Theorem 2 and Theorem 4 in [2]. The fixes are merely some appropriate substitutions of the rate and latency parameters in our Theorem 2. In Theorem 2 [2] use Theorem 2 above with the latency parameters defined as in Lemma 3 [2] (resp. for Theorem 4 [2] as defined in Lemma 5 [2]).

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